UNCUYO UNIVERSIDAD NACIONAL DE CUYO — FACULTAD DE CIENCIAS APLICADAS A LA INDUSTRIA

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Tensor tensión de Cauchy

$$\boldsymbol{T}^{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\sigma}, \qquad T_j^{(n)} = \sigma_{ij} n_i$$

$$(\boldsymbol{\sigma} - \lambda_i \boldsymbol{I})\, \boldsymbol{n}_i = \boldsymbol{0}, \qquad |\boldsymbol{\sigma} - \lambda \boldsymbol{I}| = 0$$

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

$$I_1 = \sigma_{kk}$$
$$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right)$$
$$I_3 = \det(\sigma_{ij})$$

$$\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T, \qquad \sigma'_{ij} = a_{im} a_{jn} \sigma_{mn}$$

Círculo de Mohr en tensión plana

$$\begin{aligned}
\sigma_{x'} &= \tfrac{\sigma_x+\sigma_y}{2} + \tfrac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta,\\
\sigma_{y'} &= \tfrac{\sigma_x+\sigma_y}{2} - \tfrac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta,\\
\tau_{x'y'} &= - \tfrac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta.
\end{aligned}$$

$$r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

Relaciones tensión-deformación-temperatura

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0, \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - \left(\lambda + \frac{2}{3}\mu\right)\alpha\Delta T\delta_{ij}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

	E : módulo de Young ν : coeficiente de Poisson	K : módulo de compresibilidad G : módulo de rigidez	λ : 1.er coeficiente de Lamé μ : 2º coeficiente de Lamé
(E,ν)	---	$K = \frac{E}{3(1-2\nu)}$ $G = \frac{E}{2(1+\nu)}$	$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ $\mu = \frac{E}{2(1+\nu)}$
(K,G)	$E = \frac{9KG}{3K+G}$ $\nu = \frac{3K-2G}{2(3K+G)}$	---	$\lambda = K - \frac{2G}{3}$ $\mu = G$
(λ,μ)	$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ $\nu = \frac{\lambda}{2(\lambda+\mu)}$	$K = \lambda + \frac{2\mu}{3}$ $G = \mu$	---

Tensión de Von Mises σ_{VM}

$$\sigma_{VM} = \sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}},$$

donde $\sigma_1, \sigma_2, \sigma_3$, son la tensiones principales.

Energía de deformación en cuerpos elásticos

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV.$$

Sistema lineal-elástico

$$\delta_i = \frac{\partial U}{\partial P_i}, \qquad y \qquad \phi_i = \frac{\partial U}{\partial M_i},$$

Estados planos en coordenadas cartesianas

$$\varepsilon_x = \frac{1}{\widetilde{E}}(\sigma_x - \widetilde{\nu}\sigma_y), \qquad \sigma_x = \frac{\widetilde{E}}{1-\widetilde{\nu}^2}(\varepsilon_x - \widetilde{\nu}\varepsilon_y),$$
$$\varepsilon_y = \frac{1}{\widetilde{E}}(\sigma_y - \widetilde{\nu}\sigma_x), \qquad \sigma_y = \frac{\widetilde{E}}{1-\widetilde{\nu}^2}(\varepsilon_y - \widetilde{\nu}\varepsilon_x),$$

	Tensión plana	Deformación plana
$\widetilde{\nu}$	ν	$\frac{\nu}{1-\nu}$
$\widetilde{E}$	E	$\frac{E}{1-\nu^2}$
σ_z	0	$\frac{\nu E}{1-\nu^2}(\varepsilon_x - \varepsilon_y)$
ε_z	$-\frac{\nu}{E}(\sigma_x + \sigma_y)$	0

Función de tensiones de Airy

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial y \partial x}.$$

$$\nabla^4\phi = \phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = 0$$

Estados planos en coordenadas polares

$$\sigma_{r,r} + \frac{1}{r}\tau_{r\theta,\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0,$$
$$\frac{1}{r}\sigma_{\theta,\theta} + \tau_{r\theta,r} + \frac{2\tau_{r\theta}}{r} = 0.$$

$$\varepsilon_r = u_{,r} \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}v_{,\theta} \quad \gamma_{r\theta} = \frac{1}{r}u_{,\theta} + v_{,r} - \frac{v}{r}$$

Estado plano de tensiones

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta),$$
$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r),$$
$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}.$$

Distribuciones de tensión axisimétricas

$$\phi = C_1 + C_2 \log r + C_3 r^2 + C_4 r^2 \log r,$$
$$\sigma_r = \frac{1}{r}\phi_{,r} = \frac{C_2}{r^2} + 2C_3 + C_4(2\log r + 1),$$
$$\sigma_\theta = \phi_{,rr} = -\frac{C_2}{r^2} + 2C_3 + C_4(2\log r + 3).$$

Cilindro de pared gruesa sometido a presión uniforme $C_4 = 0$

$$\sigma_r(r=a) = -p_i, \quad \sigma_r(r=b) = -p_e.$$

$$u = \frac{2(1-\nu)}{E} C_3 r - \frac{(1+\nu)}{E}\frac{C_2}{r},$$
$$\sigma_r = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} + \frac{a^2 b^2 (p_e - p_i)}{r^2(b^2 - a^2)},$$
$$\sigma_\theta = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} - \frac{a^2 b^2 (p_e - p_i)}{r^2(b^2 - a^2)}.$$

Pequeños agujeros circulares en placas tensionadas

$$\sigma_r = \sigma_\infty \left[1 - \left(\frac{R}{r}\right)^2\right], \quad \sigma_\theta = \sigma_\infty \left[1 + \left(\frac{R}{r}\right)^2\right].$$

Discos rotantes, velocidad angular ω

$$\sigma_r = \frac{3+\nu}{8}\rho\,\omega^2 \left(a^2 + b^2 - \frac{a^2 b^2}{r^2} - r^2\right),$$
$$\sigma_\theta = \frac{3+\nu}{8}\rho\,\omega^2 \left(a^2 + b^2 + \frac{a^2 b^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2\right).$$

Cilindros rotantes, velocidad angular ω

$$\sigma_r = \frac{3+\nu}{8}\rho\,\omega^2 \left(a^2 + b^2 - \frac{a^2 b^2}{r^2} - r^2\right),$$
$$\sigma_\theta = \frac{3+\nu}{8}\rho\,\omega^2 \left(a^2 + b^2 + \frac{a^2 b^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2\right),$$
$$u = r\varepsilon_\theta = \frac{r}{E}\Big[\sigma_\theta - \nu(\sigma_r + \sigma_z)\Big].$$

Deformación plana $\varepsilon_z = 0$

$$\sigma_z = \frac{3-2\nu}{4(1-\nu)}\nu\rho\,\omega^2 \left(a^2 + b^2 - \frac{2r^2}{3-2\nu}\right).$$

Deformación plana generalizada $\varepsilon_z = cte$

$$\int_0^{2\pi}\int_a^b \sigma_z\, r\, dr\, d\theta = 0,$$

$$\varepsilon_z = -\frac{\nu\rho\,\omega^2}{2E}(a^2 + b^2),$$
$$\sigma_z = \frac{\nu\rho\,\omega^2}{4(1-\nu)}(a^2 + b^2 - 2r^2).$$

Discos rotantes de espesor variable, $h = cr^{-\beta}$, c y β constantes.

$$\phi = C_1 r^{q_1} + C_2 r^{q_2} - \frac{3+\nu}{8-(3+\nu)\beta} c\,\rho\,\omega^2 r^{3-\beta},$$
$$\sigma_r = \frac{C_1}{c} r^{q_1+\beta-1} + \frac{C_2}{c} r^{q_2+\beta-1} - \frac{(3+\nu)\rho\,\omega^2 r^2}{8-(3+\nu)\beta},$$
$$\sigma_\theta = \frac{C_1}{c} q_1 r^{q_1+\beta-1} + \frac{C_2}{c} q_2 r^{q_2+\beta-1} - \frac{(1+3\nu)\rho\,\omega^2 r^2}{8-(3+\nu)\beta}.$$

Discos delgados con temperatura no uniforme

$$\sigma_r = \alpha E \frac{1}{r^2}\left[\frac{r^2 - a^2}{b^2 - a^2}\int_a^b T r\, dr - \int_a^r T r\, dr\right],$$
$$\sigma_\theta = \alpha E \frac{1}{r^2}\left[\frac{r^2 + a^2}{b^2 - a^2}\int_a^b T r\, dr + \int_a^r T r\, dr - T\, r^2\right],$$
$$u = \frac{\alpha}{r}\left[\frac{r^2(1-\nu) + a^2(1+\nu)}{b^2 - a^2}\int_a^b T r\, dr + (1+\nu)\int_a^r T r\, dr\right].$$

Cilindros largos con temperatura no uniforme

$$\sigma_r = \frac{\alpha E}{1-\nu}\frac{1}{r^2}\left[\frac{r^2 - a^2}{b^2 - a^2}\int_a^b T r\, dr - \int_a^r T r\, dr\right],$$
$$\sigma_\theta = \frac{\alpha E}{1-\nu}\frac{1}{r^2}\left[\frac{r^2 + a^2}{b^2 - a^2}\int_a^b T r\, dr + \int_a^r T r\, dr - T r^2\right],$$
$$u = r\varepsilon_\theta = \frac{r}{E}\Big[\sigma_\theta - \nu(\sigma_r + \sigma_z)\Big] + r\alpha T.$$

Extremo fijo, $\varepsilon_z = 0$:

$$\sigma_z = \frac{\alpha E}{1-\nu}\left[\frac{2\nu}{b^2 - a^2}\int_a^b T r\, dr - T\right],$$
$$u = \frac{1+\nu}{1-\nu}\frac{\alpha}{r}\left[\frac{(1-2\nu)r^2 + a^2}{b^2 - a^2}\int_a^b T r\, dr + \int_a^r T r\, dr\right].$$

Extremo libre, $\varepsilon_z = cte$:

$$\varepsilon_z = \frac{2\alpha}{b^2 - a^2}\int_a^b T r\, dr,$$
$$\sigma_z = \frac{\alpha E}{1-\nu}\left[\frac{2}{b^2 - a^2}\int_a^b T r\, dr - T\right],$$
$$u = \frac{1+\nu}{1-\nu}\frac{\alpha}{r}\left[\frac{\frac{1-3\nu}{1+\nu}r^2 + a^2}{b^2 - a^2}\int_a^b T r\, dr + \int_a^r T r\, dr\right].$$

Torsión

Eje sólido circular

$$\sigma_x = \sigma_y = \sigma_z = \tau_{rz} = \tau_{r\theta} = 0, \quad \tau_{\theta z} = G\gamma_{\theta z} = Gr\frac{d\phi}{dz}$$

$$M_t = \int_A r(\tau_{\theta z}\ dA) = G\frac{d\phi}{dz}\int_A r^2\ dA = G\frac{d\phi}{dz}I_z,$$

donde $I_z = \int_A r^2\ dA$.

$$\frac{d\phi}{dz} = \frac{M_t}{GI_z}, \quad \phi = \int_L \frac{M_t}{GI_z}dz = \frac{M_t L}{GI_z}$$

$$\tau_{\theta z} = \frac{M_t\, r}{I_z}$$

Eje hueco circular

$$I_z = \frac{\pi r_0^4}{2}\left(1 - \frac{r_i^4}{r_0^4}\right) = \frac{\pi d_0^4}{32}\left(1 - \frac{d_i^4}{d_0^4}\right)$$

Energía de deformación torsional

$$\begin{aligned} U &= \frac{1}{2}\int_V \frac{1}{G}\left(\frac{M_t r}{I_z}\right)^2 dV = \frac{1}{2}\int_L \frac{M_t^2}{G\, I_z^2}dz \int_A r^2\, dA \\ &= \frac{1}{2}\int_L \frac{M_t^2}{G\, I_z}dz \end{aligned}$$